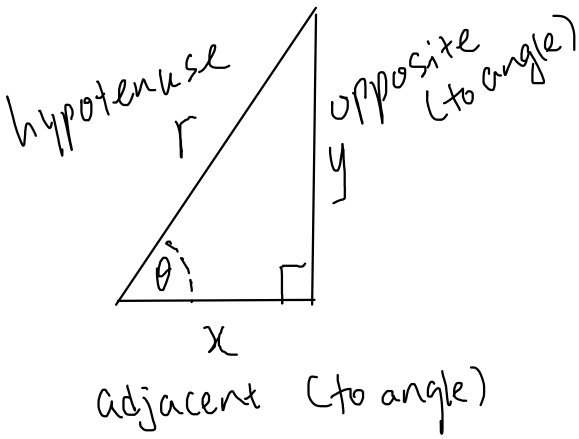
Trigonometric Functions 1

Dr. K. M. Hock



$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$

$\tan\theta = \frac{y}{x}$

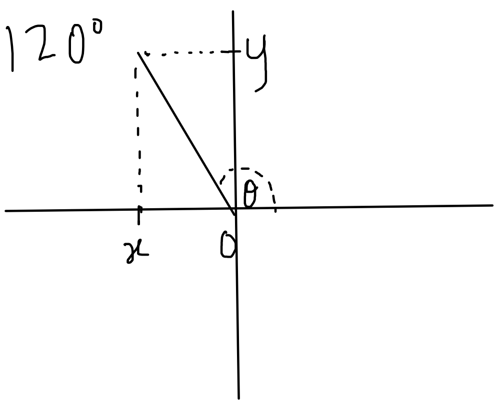
$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$

$\sec\theta = \frac{1}{\cos\theta}$

$\cot\theta = \frac{1}{\tan\theta}$

What if $\theta > 90^\circ$?

e.g. $\theta = 120^\circ$



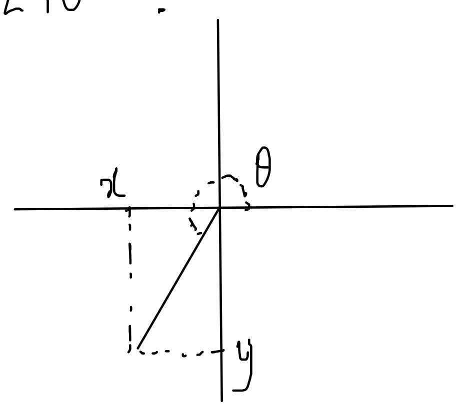
Same, but x, y can be negative.

$\sin\theta = \frac{y}{r}$

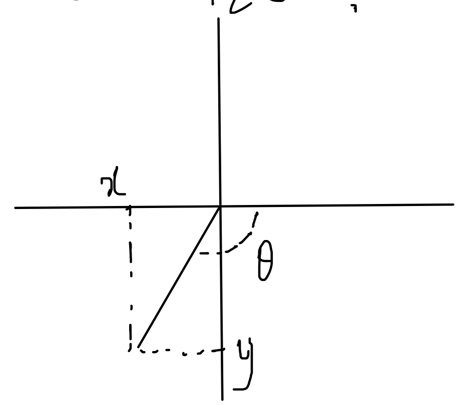
$\cos\theta = \frac{x}{r}$

$\tan\theta = \frac{y}{x}$

or $\theta = 240^\circ$?



or $\theta = -120^\circ$?



Same formulae for any angle:

- just sketch on x, y axis, and
- draw angle from +ve x direction, with
- anti-clockwise for +ve angle.

Trigonometric Functions 2

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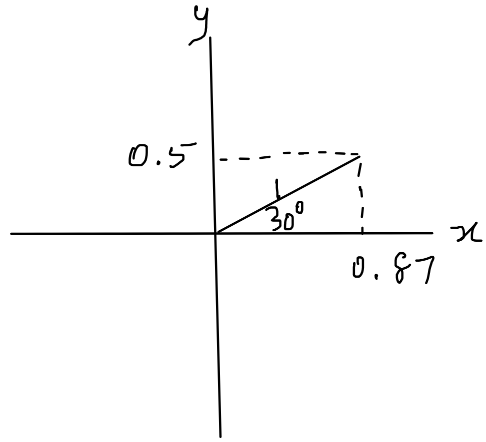
e.g.

radians

$\sin 30^\circ = \dfrac{0.5}{1} = \sin \dfrac{\pi}{6}$

$\cos 30^\circ = \dfrac{0.87}{1} = \cos \dfrac{\pi}{6}$

$\tan 30^\circ = \dfrac{0.5}{0.87} = \tan \dfrac{\pi}{6}$



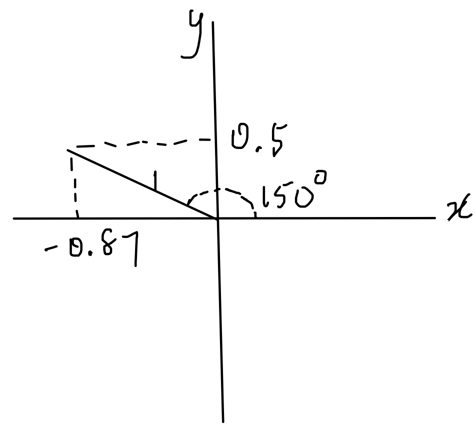
e.g.

radians

$\sin 150^\circ = \dfrac{0.5}{1} = \sin \dfrac{5\pi}{6}$

$\cos 150^\circ = \dfrac{-0.87}{1} = \cos \dfrac{5\pi}{6}$

$\tan 150^\circ = \dfrac{0.5}{-0.87} = \tan \dfrac{5\pi}{6}$

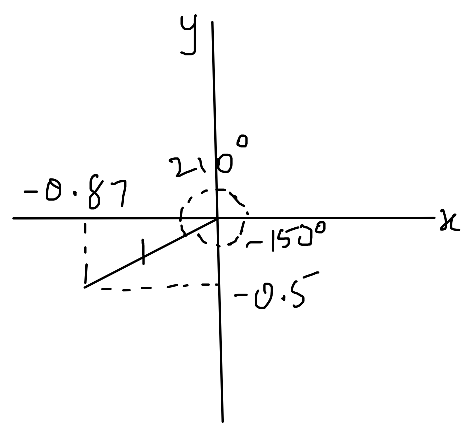


e.g.

$\sin 210^\circ = \dfrac{-0.5}{1} = \sin(-150^\circ)$

$\cos 210^\circ = \dfrac{-0.87}{1} = \cos(-150^\circ)$

$\tan 210^\circ = \dfrac{-0.87}{-0.5} = \tan(-150^\circ)$

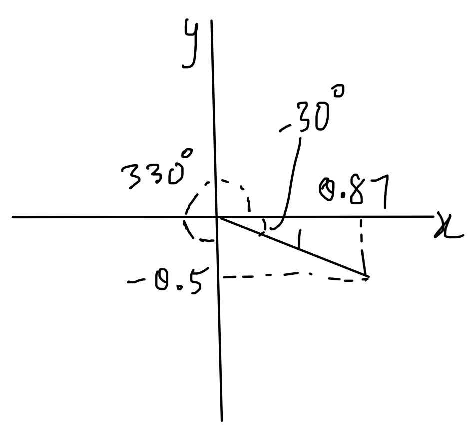


e.g.

$\sin 330^\circ = \dfrac{-0.5}{1} = \sin(-30^\circ)$

$\cos 330^\circ = \dfrac{0.87}{1} = \cos(-30^\circ)$

$\tan 330^\circ = \dfrac{0.87}{-0.5} = \tan(-30^\circ)$



Principal values of $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$.

Inverse Functions

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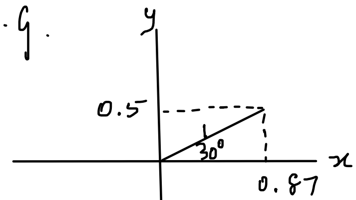
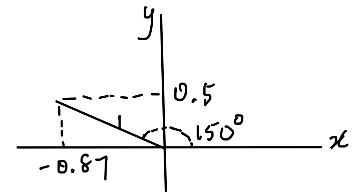
e.g. $\sin 30^\circ = \frac{0.5}{1} = \sin 150^\circ$





So if $\sin\theta = 0.5$, $\theta = 30^\circ$ or 150° ?

For inverse function in trigo., need to define a range.

e.g. Let $f(\theta) = \sin\theta$.

Write inverse as $f^{-1}(\theta) = \sin^{-1}\theta$.

$$\boxed{\text{So } \sin^{-1}\theta \neq \frac{1}{\sin\theta} \; !}$$

e.g. Find $\sin^{-1} 0.5$.

For inverse function of $\sin\theta$, range is $-90^\circ \le \theta < 90^\circ$.

We know $\sin 30^\circ = \sin 150^\circ = 0.5$

But we choose $\sin^{-1} 0.5 = 30^\circ$ $\because$ it is in

An answer in the range defined is called principal value.

	range of principal values
$y = \sin^{-1} x$	$-90^\circ \le y \le 90^\circ$
$y = \cos^{-1} x$	$0^\circ \le y \le 180^\circ$
$y = \tan^{-1} x$	$-90^\circ \le y \le 90^\circ$

Exact values for the trigonometric functions for special angles (30°, 45°, 60°) or ($\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$).

Special Angles

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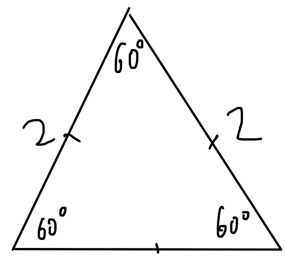
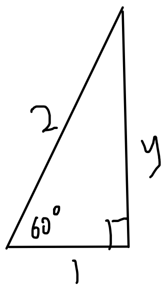
e.g. Take an equilateral Δ.

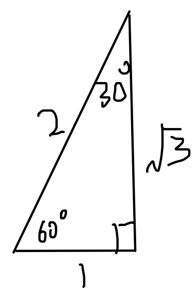




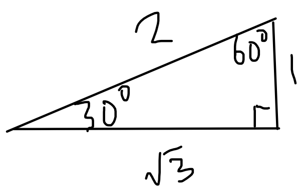
Chop it in half.

Use Pythagoras theorem: $y = \sqrt{2^2 - 1^2} = \sqrt{3}$

Then find
$$\sin 60^\circ = \frac{\sqrt{3}}{2}$$
$$\cos 60^\circ = \frac{1}{2}$$
$$\tan 60^\circ = \sqrt{3}$$



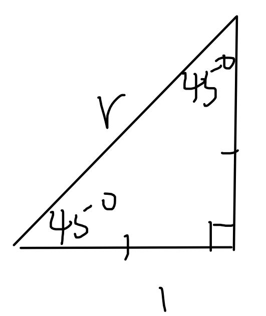
e.g.

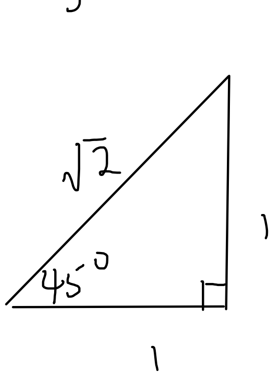


Next flip it over.

Then find
$$\sin 30^\circ = \frac{1}{2}$$
$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$
$$\tan 30^\circ = \frac{1}{\sqrt{3}}$$

e.g. Finally try this isosceles Δ.





Use Pythagoras:

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Then get
$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$
$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$
$$\tan 45^\circ = 1$$

Amplitude, periodicity and symmetries related to the sine and cosine functions

Graphs 1

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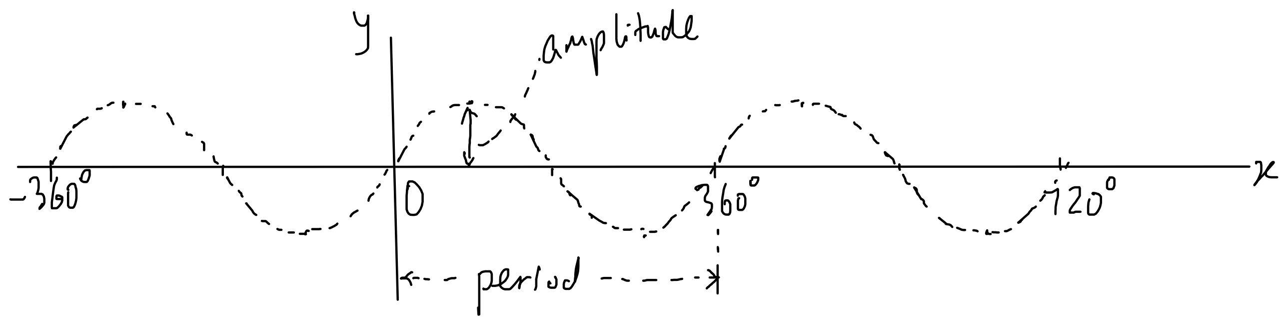
e.g.

x	$0°$	$45°$	$90°$	$135°$	$180°$	$225°$	$270°$	$315°$	$360°$
$\sin x$	0	0.71	1	0.71	0	−0.71	−1	−0.71	1

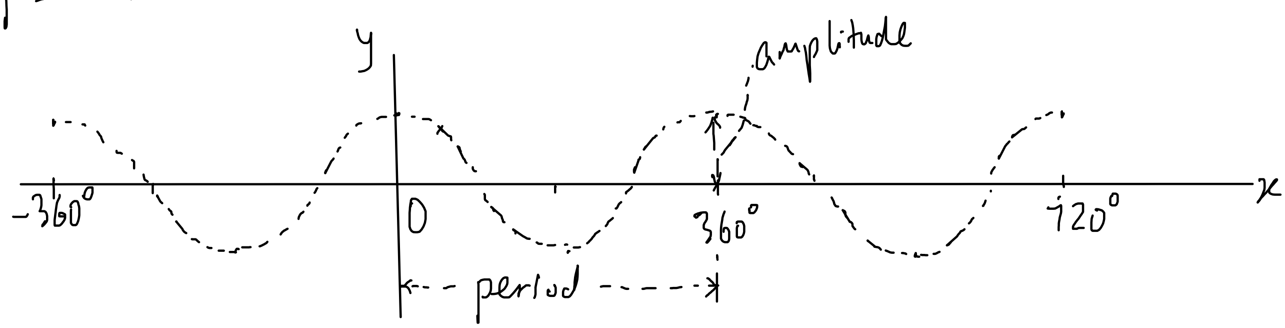


Notice : - maximum displacement → amplitude
- repeating unit → period
- symmetry → odd

e.g. $\sin(-90°) = -\sin 90°$

so inversion in 0 symmetrical.

e.g. $y = \cos x$.



- Symmetry - even e.g. $\cos(-45°) = \cos 45°$
  - reflection in y-axis symmetrical
- periodic e.g. $\cos(45° + 360°) = \cos 45°$
  - shift 360° to right - looks same

Graphs of $y = a\sin(bx) + c$, $y = a\sin\left(\frac{x}{b}\right) + c$, $y = a\cos(bx) + c$, $y = a\cos\left(\frac{x}{b}\right) + c$ and $y = a\tan(bx)$ where a is real, b is a positive integer and c is an integer.

Transforming Graphs

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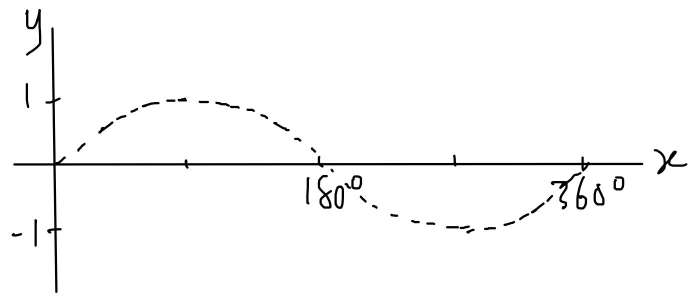
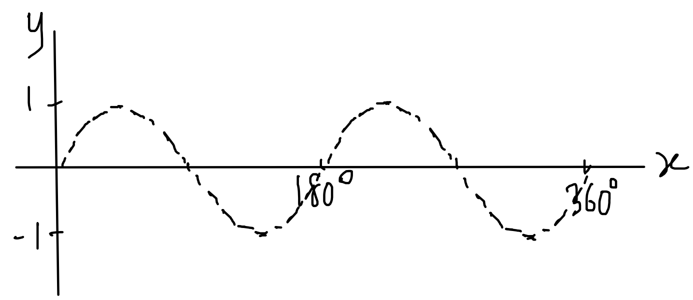
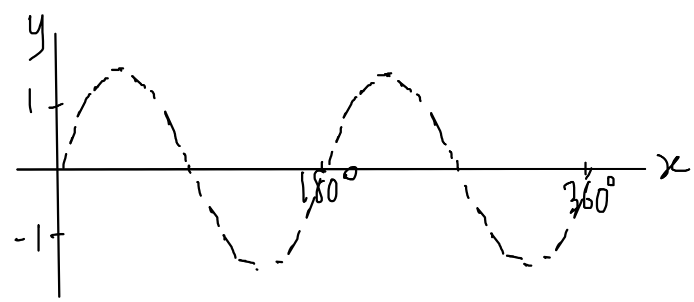
e.g. $y = \sin x$



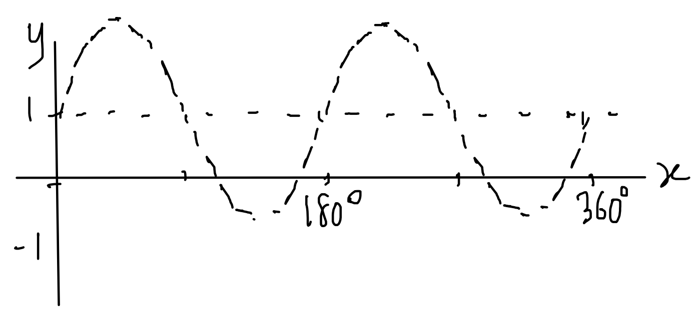
e.g. $y = \sin 2x$



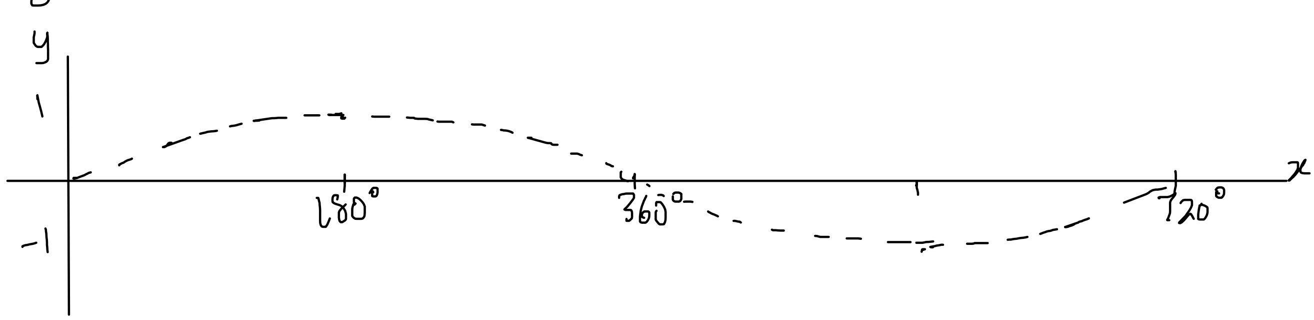
e.g. $y = 1.5\sin 2x$



e.g. $y = 1.5\sin 2x + 1$



e.g. $y = \sin\frac{x}{2}$



Graphs of $y = a\sin(bx) + c$, $y = a\sin\left(\frac{x}{b}\right) + c$, $y = a\cos(bx) + c$, $y = a\cos\left(\frac{x}{b}\right) + c$ and $y = a\tan(bx)$ where a is real, b is a positive integer and c is an integer.

Tangent Graphs

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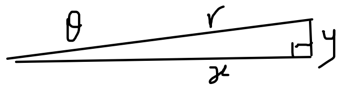
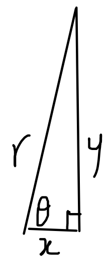
e.g. When $\theta \to 0$



$y \to 0$ and $x \to r$

So
$$\sin 0^\circ = \frac{y}{r} \to \frac{0}{r} = 0$$
$$\cos 0^\circ = \frac{x}{r} \to \frac{r}{r} = 1$$
$$\tan 0^\circ = \frac{y}{x} \to \frac{0}{x} = 0$$

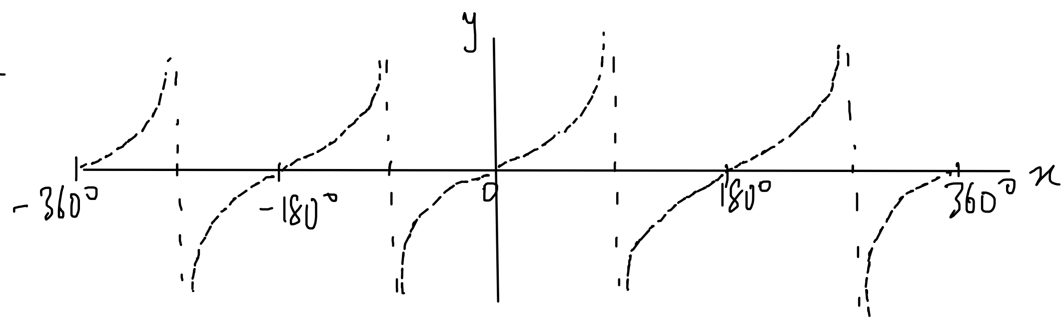
when $\theta \to 90^\circ$



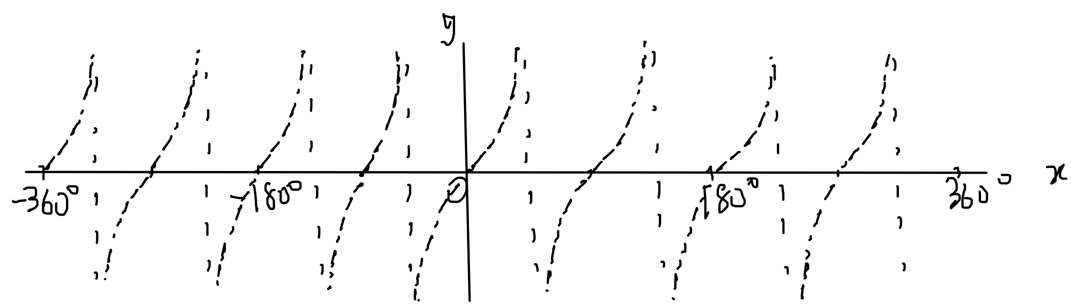
$y \to r$, $x \to 0$

So
$$\sin 90^\circ = \frac{y}{r} \to \frac{r}{r} = 1$$
$$\cos 90^\circ = \frac{x}{r} \to \frac{0}{r} = 0$$
$$\tan 90^\circ = \frac{y}{x} \to \frac{y}{0} = \infty$$

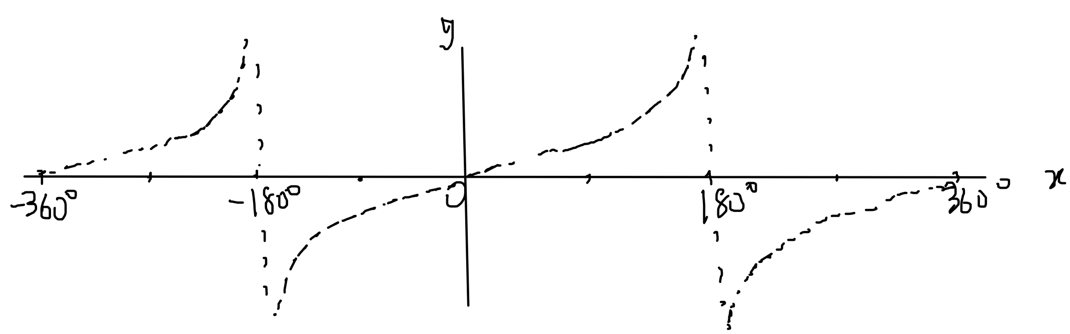
e.g. $y = \tan x$



e.g. $y = \tan 2x$



e.g. $y = \tan \frac{x}{2}$



Identities 1

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$$\frac{\sin A}{\cos A} = \tan A \qquad \because \tan A = \frac{y}{x} = \frac{y/r}{x/r} = \frac{\sin A}{\cos A}$$

$$\frac{\cos A}{\sin A} = \cot A \qquad \because \cot A = \frac{1}{\tan A}$$

$$\sin^2 A + \cos^2 A = 1 \qquad \because x^2 + y^2 = r^2 \Rightarrow \frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$\Rightarrow \sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A \qquad \because \sin^2 A + \cos^2 A = 1 \Rightarrow \frac{\sin^2 A}{\cos^2 A} + 1 = \frac{1}{\cos^2 A}$$

$$\Rightarrow \tan^2 A + 1 = \sec^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A \qquad \because \sin^2 A + \cos^2 A = 1 \Rightarrow 1 + \frac{\cos^2 A}{\sin^2 A} = \frac{1}{\sin^2 A}$$

$$\Rightarrow 1 + \cot^2 A = \operatorname{cosec}^2 A$$

Identities 2

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1. $\sin(A+B) = \sin A \cos B + \cos A \sin B$

2. $\sin(A-B) = \sin A \cos B - \cos A \sin B$

3. $\cos(A+B) = \cos A \cos B - \sin A \sin B$

4. $\cos(A-B) = \cos A \cos B + \sin A \sin B$

5. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

6. $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(Not in syllabus but useful e.g. for 2013 P2 Q4 on page 16.)

Identities 2 Proofs

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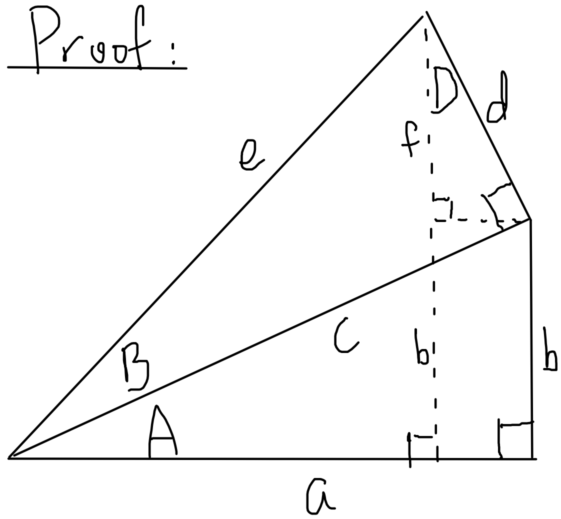
1. $\sin(A+B) = \sin A \cos B + \cos A \sin B$

Proof:



$$\sin(A+B) = \frac{f+b}{e}$$

$$= \frac{f}{e} + \frac{b}{e}$$

$$= \frac{f}{d}\cdot\frac{d}{e} + \frac{b}{c}\cdot\frac{c}{e}$$

$$= \cos D \sin B + \sin A \cos B$$

$$= \cos A \sin B + \sin A \cos B$$

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2. $\sin(A-B) = \sin A \cos B - \cos A \sin B$

Proof: Derive from 1. by letting B be −B.

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3. $\cos(A+B) = \cos A \cos B - \sin A \sin B$

Proof: Derive from 1. by setting A to $90° - A$ and using $\cos x = \sin(90° - x)$.

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5. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

Proof: Divide Identity 1 by Identity 3 and rearrange.

Identities 3

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7. $\sin 2A = 2 \sin A \cos A$

Proof:
Let $B = A$ in identity 1.

8. $\cos 2A = \cos^2 A - \sin^2 A$

Proof:
Let $B = A$ in identity 3.

9. $\cos 2A = 2 \cos^2 A - 1$

Proof:
Use $\cos^2 A + \sin^2 A = 1$ in identity 8.

10. $\cos 2A = 1 - 2 \sin^2 A$

Proof:
Use $\cos^2 A + \sin^2 A = 1$ in identity 8.

11. $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

Proof:
Let $B = A$ in identity 5.

Identities 4

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Identities: 1. $\sin(A+B) = \sin A \cos B + \cos A \sin B$

2. $\sin(A-B) = \sin A \cos B - \cos A \sin B$

Adding: $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$

Let $P = A+B$, $Q = A-B$. Then $A = \frac{P+Q}{2}$, $B = \frac{P-Q}{2}$.

$\therefore$, $\sin P + \sin Q = 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2}$.

More identities:

12 $\sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$ From above, $P \to A$, $Q \to B$

13 $\sin A - \sin B = 2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)$ Likewise by subtracting identities 1, 2.

14 $\cos A + \cos B = 2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$ By adding identities 3, 4

15 $\cos A + \cos B = -2 \sin \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)$ By subtracting identities 3, 4

the expression for $a\cos\theta + b\sin\theta$ in the form $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$

R Formula

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e.g. Express $3\cos\theta + 4\sin\theta$ in the form $R\cos(\theta+\alpha)$.

Ans. Using identity, $\cos(\theta+\alpha) = \cos\theta\cos\alpha - \sin\theta\sin\alpha$.

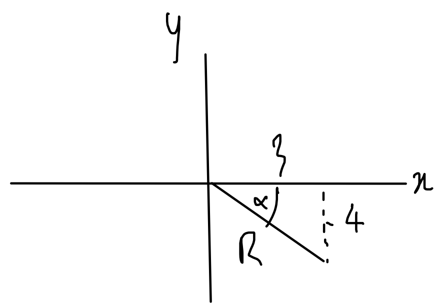
Let $3\cos\theta + 4\sin\theta = R\cos(\theta+\alpha)$.

So $3\cos\theta + 4\sin\theta = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Comparing coefficients of $\cos\theta$: $3 = R\cos\alpha$ — (1)

of $\sin\theta$: $4 = -R\sin\alpha$ — (2)

Solving



$R = \sqrt{3^2+4^2} = 5$

$\tan\alpha = -\frac{4}{3}$

$\alpha = -53.13°$

$\therefore 3\cos\theta + 4\sin\theta = 5\cos(\theta - 53.13°)$

Problem 1

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2013 P1 Q2

Express $\sin\left(\frac{7\pi}{12}\right) + \sin\left(\frac{\pi}{12}\right)$ in the form $\frac{\sqrt{k}}{2}$.

Solution. Identity: $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$.

Let $A = \frac{7\pi}{12}$, $B = \frac{\pi}{12}$. Then $\frac{A+B}{2} = \frac{\pi}{3}$, $\frac{A-B}{2} = \frac{\pi}{4}$.

$$\therefore \sin\frac{7\pi}{12} + \sin\frac{\pi}{12} = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4}$$

$$= 2 \times \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$$

$$= \sqrt{3} \times \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{6}}{2}.$$

Problem 2

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2013 P1 Q4 (i) Show that $\dfrac{(\sin A - \cos A)(1 + \sin A \cos A)}{\cos^3 A} = \tan^3 A - 1$.

(ii) Hence find the acute angle A such that $(\sin A - \cos A)(1 + \sin A \cos A) = 2\cos^3 A$.

Solution: (i). LHS $= \dfrac{\sin A + \sin^2 A \cos A - \cos A - \sin A \cos^2 A}{\cos^3 A}$

$$= \frac{\sin A + (1-\cos^2 A)\cos A - \cos A - \sin A(1-\sin^2 A)}{\cos^3 A}$$

$$= \frac{\sin A + \cos A - \cos^3 A - \cos A - \sin A + \sin^3 A}{\cos^3 A}$$

$$= \frac{-\cos^3 A + \sin^3 A}{\cos^3 A}$$

$$= -1 + \tan^3 A = \text{RHS}$$

(ii) Rearrange: $\dfrac{(\sin A - \cos A)(1 + \sin A \cos A)}{\cos^3 A} = 2$

From (i), $\tan^3 A - 1 = 2$

$$\tan^3 A = 3$$

$$\tan A = \sqrt{3}$$

$$A = 60^\circ$$

Problem 3

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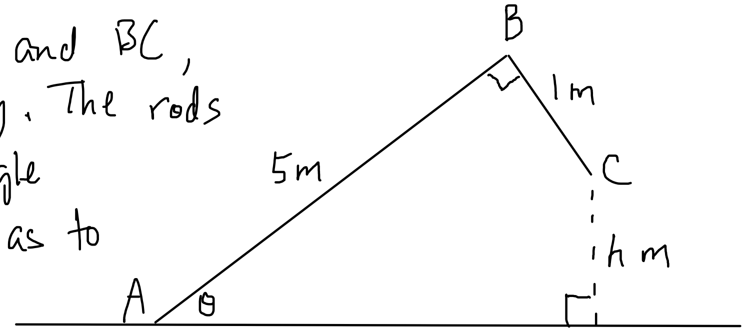
2013 P2 Q4 The diagram shows 2 rods, AB and BC, of length 5 m and 1 m respectively. The rods are fixed at B such that the angle $ABC = 90^\circ$ and hinged at A so as to rotate in a vertical plane. The rod AB makes an acute angle with the vertical ground.



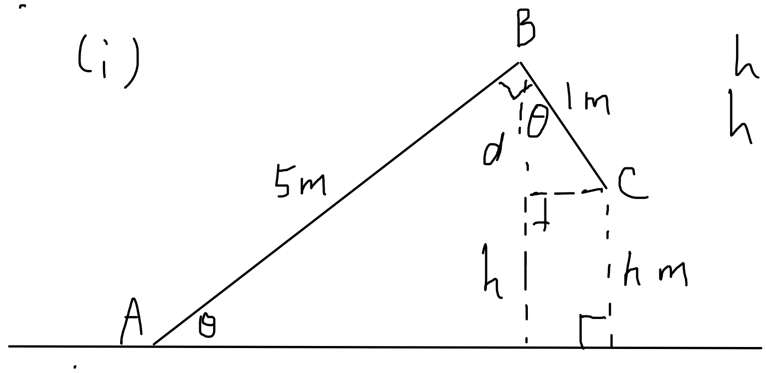
(i) Obtain an expression, in terms of θ, for h, where h m is the height of C above the ground.

(ii) Express h in the form $R\sin(\theta-\alpha)$ where $R>0$ and $0^\circ<\alpha<90^\circ$.

(iii) Find the value of θ for which C is 3 m above the ground.

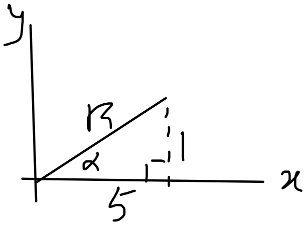
Solution.

(i)



$$h + d = 5\sin\theta$$
$$h + 1\cos\theta = 5\sin\theta$$
$$h = 5\sin\theta - \cos\theta$$

(ii) $R\sin(\theta-\alpha) = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$



$R = \sqrt{5^2+1^2} = \sqrt{26}$ $\quad \alpha = \tan^{-1}\frac{1}{5} = 11.31^\circ$

$$\therefore h = \sqrt{26}\sin(\theta - 11.31^\circ)$$

(iii)
$$\sqrt{26}\sin(\theta - 11.31^\circ) = 3$$
$$\theta - 11.31^\circ = \sin^{-1}\frac{3}{\sqrt{26}} = 36.04^\circ$$
$$\theta = 47.35^\circ$$

Problem 4

Dr. K. M. Hock

2013 P2 Q6 Given that $y = p + q\sin 3x$, where p and q are positive integers.

(i) state the period of y.

Given that the maximum and minimum values of y are 6 and -2 respectively, find

(ii) the amplitude of y, (iii) the value of p and of q.

Using the values of p and q found in part (iii),

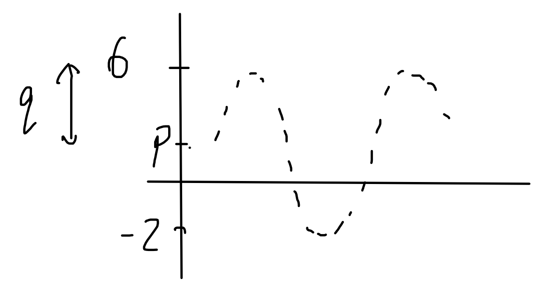
(iv) find, in degrees, the smallest positive value of x for which $y=0$,

(v) sketch the graph of y for $0^\circ \le x \le 240^\circ$.

Solution.

(i) $\dfrac{360^\circ}{3} = 120^\circ$

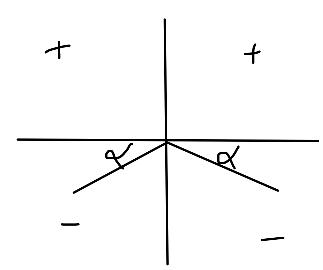
(ii) $\dfrac{6-(-2)}{2} = 4$



(iii) $q = 4$ from (ii)

$p = \dfrac{6+(-2)}{2} = 2.$

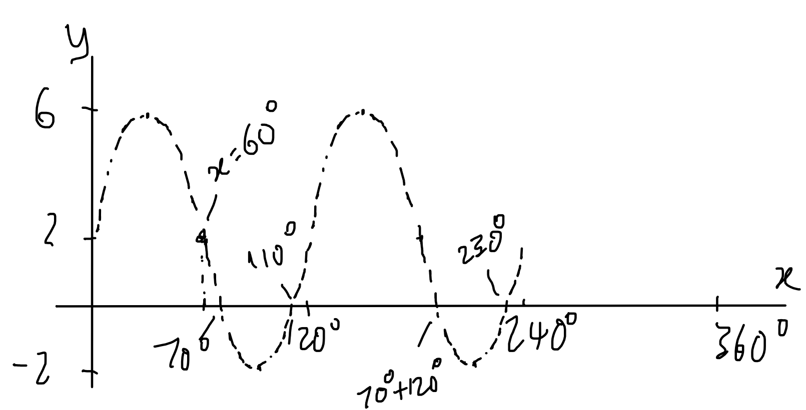
(iv)
$$y = 2 + 4\sin 3x$$
$$0 = 2 + 4\sin 3x$$
$$\sin 3x = -\frac{1}{2}$$
$$3x = 180^\circ + \alpha$$
$$= 210^\circ$$
$$x = 70^\circ$$



basic angle: $\sin\alpha = \frac{1}{2}$

$\alpha = 30^\circ$

(v)



3 cycles in 360°

2 in 240°